1 - Create one example for each of the following financial situations.

a) Use the concept of short-selling with one European put option and a stock for the purpose of Speculation. Show a proper graph for this example.

b) Use the concept of short-selling with one European call option, one European put option and a stock for the purpose of Hedging. Show a proper graph for this example.

2 - Assume you will observe the price of a stock for two consecutive days. The stock will be worth $\{80, 110\}$ on day 1 with probabilities $1/3$ and $2/3$ respectively, and $\{90, 100, 120\}$ on day 2 with probabilities $1/4, 1/2$ and $1/4$ respectively. Prices on day 1 and day 2 are independent.

a) Describe the sample space for this experiment: $\Omega$. Compute the probabilities of the outcomes in $\Omega$.

b) Define the random variable $X$ as the maximum price between days one and two. Describe the values of $X$ and their probabilities.

c) Describe four events in the Borel $\sigma$-algebra generated by $X$: $\beta_X$.

d) Define the function $Y$ as follows: $Y(\omega) = 0$ for all $\omega \in \Omega$ such that price on day 2 is less than 95, $Y(\omega) = 25$ for all $\omega \in \Omega$ such that price on day 2 is greater or equal than 95. Is $Y$ a proper random variable in $\mathcal{F}_1 = \{\{(80, 90), (80, 100), (80, 120)\}, \{(110, 90), (110, 100), (110, 120)\}, \emptyset, \Omega\}$?. Explain

3 - Let the prices of two stocks $(X, Y)$ after one year be as follows: $\Omega = \{\{(70, 70)\}, \{(70, 100)\}, \{(90, 100)\}, \{(100, 90)\}\}$. Take the $\sigma$-algebra as $\mathcal{F}_\Omega$ and the uniform probability measure.

a) Describe the random variables $X$ and $Y$, i.e. their values and probabilities.

b) Let $B = \{X \leq 90\}$, compute $P(X \mid B)$, $P(Y \mid B)$ and $E[Y \mid 1_B]$ .

c) Describe the random variables $E[X \mid Y]$ and $E[Y \mid X]$ i.e. values and probabilities.

d) Compute $E[(X+Y-180)^+]$ using explicitly the Tower Property. i.e. compute $E[(X+Y-180)^+ \mid Y=y]$ for all $y$.

4 - Assume a single period Binomial Model with two independent stocks. For stock $i=1,2$ let $S_i(0)=1$, $u_i=1.2$ and $d_i=0.8$ be the size of the up and down movements. Let $p_1=0.4$, $p_2=0.3$ be the probabilities of an up movement for stocks 1 and 2 respectively, $T=1$, $r=0.1$ and $B(0)=1$.

a) Describe the sample space $\Omega$ and the probabilities $P$ for the triple $(S_1(1), S_2(1), B(1)))$.

b) Define the portfolio $X(t)=S_1(t)+S_2(t)-B(t)$. Describe the random variable $X(1)$ (possible values as a function of $\Omega$ and probabilities).

c) Compute $E[X(1)]$, $V[X(1)]$ and $E[X(1) \mid S_2(1)=0.7]$.

d) Assume an investor speculates that the second stock will increase in value, what would the expected value of the portfolio be if the investor is correct?. Compare to part c.

5 - Consider the following 3-by-4 Arrow-Debrew model (three assets $S_i$, $i=1.,,,3$, four states, $\omega_j$ , $j=1,...,4$):

$$D=\begin{pmatrix} 100 & 90 & 80 & 70 \\ 100 & 110 & 120 & 120 \\ 70 & 90 & 80 & 90 \end{pmatrix}$$

Assume the probability are as follows, $P(\omega_1)=P(\omega_2)=P(\omega_3)=1/5$, $P(\omega_4)=2/5$.

a) Find $E[S_1(1)]$ and $V[S_1(1)]$.

b) What are the probabilities of the events: $B_1=\{S_1(1)>90, S_2(1)<110\}$ and $B_2=\{S_1(1)\geq 100, S_2(1)<110, S_3(1)=90\}$.

c) Define $Y=\min\limits_{1\leq i\leq 3}\{S_i(1)-100\}$, describe the random variable $Y$.

d) Which one(s) of the following payoff are attainable:

$Y(1)=(S_1(1)+S_2(1)-S_3(1)-100)^+$ ;

$Y(1,\omega_1)=0$, $Y(1,\omega_2)=20$, $Y(1,\omega_3)=10$, $Y(1,\omega_4)=10$;

$Y(1)=(S_1(1)-90)^+ + \frac{S_2(1)}{2} - \frac{S_3(1)}{2}$.

Explain and show the replicating portfolio.

e) Is this model redundant?. If yes then remove the redundant stock.

6 - Assume a Bond with $r=0$ and initial value 1, three stocks $(N=3)$ with initial values 1 and two possible movements per stock $(K=2$, up $u$, down $d)$, five market states, $R=5$, $\omega_1=\{d,d,d\}$, $\omega_2=\{d,d,u\}$, $\omega_3=\{d,u,d\}$, $\omega_4=\{u,d,d\}$ and $\omega_5=\{u,u,d\}$ with probabilities $1/4, 1/4, 1/4, 1/8$ and $1/8$ respectively. Moreover assume the strategy: $\delta=(1,0.5,0.5,-0.5)$.

a) Describe the probabilities $P$ for each random variable $S_1(1)$, $S_2(1)$, $S_3(1)$ and for the pair $(S_1(1), S_2(1))$.

b) What would the self-financing condition be for this market model? Describe the wealth process $X(1)$ and the gains $G$.

c) Assume $u = 1.2$ and $d = 0.8$, construct the cash-flow matrix.

7) Consider the 3-period Binomial Model as an stochastic process $\{S(t)\}$ with $t = 0, 1, 2, 3$ on a filtered probability space $(\Omega, \mathcal{F}_\Omega, \beta_t, P)$.

a) Assume $u = 1/d$, describe the sample space $\Omega$, the probabilities $P$ and the elements $\beta_1$ and $\beta_2$ in the filtration.

b) Plot a few sample paths of the process.

c) Write $S(2)$ and $S(3)$ as functions of $\omega$ and explicitly identify its possible values $(s(i, K_i), i = 2, 3)$. Describe the probabilities for $S(2)$ and $S(3)$.

d) Assume $S(0) = 100$, $u = 1.25$, $d = 1/u$ and $p = 3/4$ and define the process $M(t) = \max_{0 \le s \le t} S(s) - \min_{0 \le s \le t} S(s)$. Express $M(3, \omega)$ as a function of the path $\omega$. Compute the probabilities for $M(3)$.

e) Compute $E[M(3) \mid \beta_2]$ (Express your answer as a function of the path.)